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LETTERS TO THE EDITOR

PREDICTING LOCALIZATION VIA LYAPUNOV EXPONENT STATISTICS

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1. INTRODUCTION

The localization factor, which was first developed in solid state physics [1–6], has often been used to quantify vibration localization in mono-coupled nearly periodic structures [7–12]. The localization factor is the average exponential decay rate of the vibration amplitudes, measured from one substructure to the next. For multi-coupled structures, a traditional localization factor cannot be found, but one may compute the Lyapunov exponents of the system wave transfer matrix [4, 8]. The Lyapunov exponents are analogous to the localization factor, since they provide a measure of the spatial amplitude decay rates for the multiple wave types. In fact, for the mono-coupled case, the largest Lyapunov exponent is equivalent to the localization factor [13, 14]. Lyapunov exponents may be calculated numerically using an efficient algorithm developed by Wolf *et al*. [15].

Spatial wave decay in a nearly periodic structure, however, is not necessarily due to disorder. The wave decay can be caused by other mechanisms, such as off-resonance or damping. We are thus posed with the problem that, although we can quantify the decay rate, we do not know if this value truly indicates localization. Indeed, a localization factor or Lyapunov exponent is sometimes greatest in frequency regions where there is, in fact, little localization. The crucial distinction between localization due to disorder and wave decay caused by other mechanisms is that localization is a *confinement*, rather than a dissipation or attenuation, of the vibration energy. This energy confinement can lead to higher response amplitudes in a disordered system than would be found in its ordered counterpart. It is therefore of practical interest to identify the frequency regions in which localization is the primary source of decay.

In reference [16], Lyapunov exponents were shown to provide a valuable tool for analyzing wave decay in multi-coupled systems. Here, the work of reference [16] is extended: we propose that localization may be predicted by calculating the statistics of the Lyapunov exponents. Since a numerical Lyapunov exponent is actually an average of the values computed at each iteration of the algorithm, by considering only the mean we have discarded valuable information. The standard deviation of the iterate values can be used as well. If the decay is due mostly to damping or off-resonance, then each iterate value will be approximately the same, and the standard deviation will be small relative to the mean. If the decay is due to disorder, then the iterate values will vary significantly, and the larger standard deviation indicates localization. Thus, by considering both the mean and standard deviation, we can systematically identify frequency ranges where localization occurs.

Very few studies have examined the statistics of Lyapunov exponents. Cha and Morganti [17] investigated the mean, and probability density of the localization factor for the mono-coupled system considered here. However, they used this information to make correct inferences about the rate of exponential decay of a typical system, while we use the standard deviation for a different purpose. Cusumano and Lin [18] calculated the

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covariance matrices for the Lyapunov *vectors* of a non-linear system in order to determine modal interaction. We adopt a similar approach here by examining the covariance of the Lyapunov *exponents* as an indicator of wave conversion [19, 20].

This work is a follow-up to reference [16], which the interested reader may consult for background on the theory of Lyapunov exponents or for more detailed discussion of the two example systems revisited here. In the present study, we examine the statistics of the Lyapunov exponents for these example systems, and we draw conclusions regarding the use of a Lyapunov exponent analysis.

2. LYAPUNOV EXPONENTS

The algorithm of Wolf *et al*. [15] for computing the first (largest) Lyapunov exponent, γ_1, which is the localization factor of a mono-coupled system, is as follows:

$$\gamma_1 \approx \frac{1}{N}\sum_{i=1}^{N} \ln \left\| W_i \frac{v_{i-1}}{\|v_{i-1}\|} \right\|, \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm, W_i is the wave transfer matrix for the ith substructure, and v_{i-1} is the wave amplitude vector at the junction of substructures $i-1$ and i. Note that the final Lyapunov exponent is simply the average of N iterate values. The result of this algorithm converges to the exact first Lyapunov exponent as N becomes large.

A similar algorithm may be used to find the other Lyapunov exponents for a multi-coupled system. There exists one non-negative Lyapunov exponent for each coupling type in a nearly periodic system. Accordingly, in the following examples, one Lyapunov exponent is considered for the mono-coupled case, and two are considered for the bi-coupled case.

As an example of a mono-coupled system, we consider a chain of one-degree-of-freedom oscillators coupled by linear springs, as shown in Figure 1(a). Each oscillator has mass m and structural damping factor δ, and each coupling spring has stiffness k_c. An arbitrary ith oscillator is considered to have a random stiffness, $k_i = k(1+f_i)$, where k is the nominal

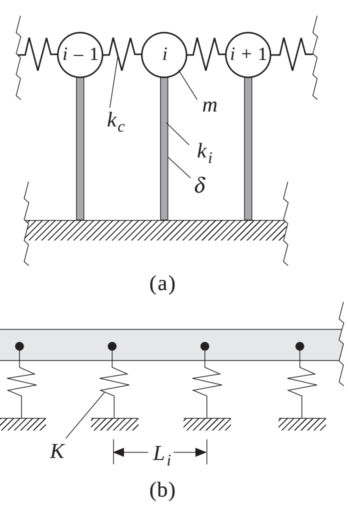

Figure 1. (a) An example of a mono-coupled system: a chain of oscillators coupled by linear springs. Each oscillator is considered to have a one-degree-of-freedom tip deflection. (b) An example of a bi-coupled system: a multi-span beam pinned to elastic supports. Each support is modelled by a linear spring.

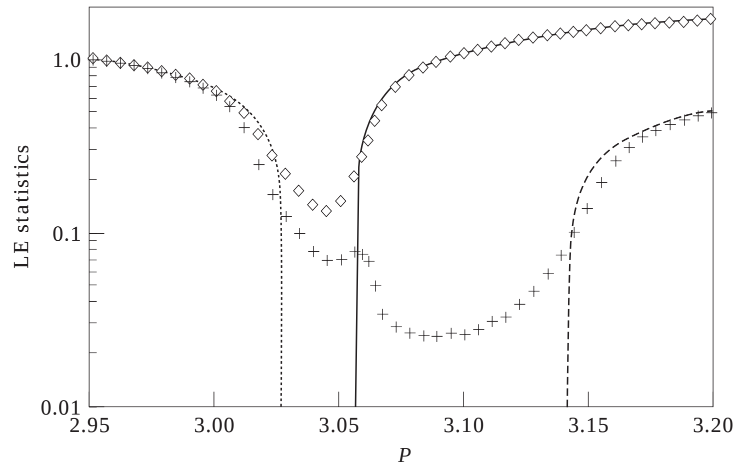

Figure 2. The Lyapunov exponents for the multi-span beam with $\bar{K} = 50$. Ordered case: $\cdots$, $\gamma_1 = \gamma_2$; ——, γ_1; ----, γ_2. Disordered case, $S = 0{\cdot}01$, $N = 50\,000$ iterations: $\diamond$, γ_1; $+$, γ_2.

stiffness, and f_i is the disorder taken from a uniformly distributed random variable with mean zero and standard deviation S. For the ordered case, of course, $f_i \equiv 0$. We define the coupling strength, R, and the dimensionless frequency, $\bar{\omega}$, as

$$R \equiv k_c/k, \qquad \bar{\omega} \equiv \omega/\sqrt{k/m}, \tag{2}$$

where ω is the frequency. This mono-coupled system is discussed in detail in reference [21].

As an example of a multi-coupled system, we present the structure shown in Figure 1(b). This is an undamped multi-span beam undergoing transverse bending motion, pinned at flexible supports that are modelled by linear springs of stiffness K. Since the beam has a vertical displacement and a rotation at each support, the substructures are coupled through two coupling co-ordinates, so this is a bi-coupled structure.

Each span has length L for the ordered case. Disorder is introduced by allowing the length of span i to be $L_i = L(1 + f_i)$, where f_i is a value taken from a uniformly distributed random variable with mean 0 and standard deviation S. The beam is modelled by Bernoulli–Euler theory; it has Young's modulus E, cross-sectional inertia I, and mass per unit length μ. We use the dimensionless parameters

$$\bar{K} \equiv KL^3/2EI, \qquad P \equiv \sqrt{\omega/\sqrt{EI/\mu L^4}}, \tag{3}$$

where $\bar{K}$ is the dimensionless support stiffness, and P is a dimensionless frequency parameter. We refer the interested reader to reference [22], where this system is considered in detail.

Let us now consider the Lyapunov exponents of an ordered multi-span beam with $\bar{K} = 50$. The first (γ_1) and second (γ_2) Lyapunov exponents, which are shown versus the frequency parameter P in Figure 2, indicate four distinct frequency regions. For $P < 3{\cdot}027$, $\gamma_1 = \gamma_2 \neq 0$, so the waves are in a *complexband*. For $3{\cdot}027 < P < 3{\cdot}056$, $\gamma_1 = \gamma_2 = 0$, so both wave types are in a passband (*double passband*). For $3{\cdot}056 < P < \pi$, wave type *I* lies in a stopband while the wave type *II* belongs still to a passband (*stopband–passband*). For $P > \pi$, both wave types are in stopbands (*double stopband*).

In the case of an ordered system, γ_1 (γ_2) is the *exact* spatial amplitude decay rate of wave type I (II). For a disordered system, in which the wave transfer matrix for each substructure is random, the Lyapunov exponents indicate wave decay which may be due to attenuation and/or localization.

In Figure 2, the Lyapunov exponents calculated using the Wolf algorithm (50 000 iterations) for a disordered case ($S = 0{\cdot}01$) are also shown. In reference [16], the authors examined simulations of right-travelling wave amplitudes, and concluded the following. In frequency regions in which the Lyapunov exponents of the disordered system are close to those of the ordered system (complexband and double stopband), the decay is due primarily to off-resonance effects. In the double passband, the wave decay is due to localization, and the two waves have similar decay rates. In the stopband–passband region, the two waves have distinct decay rates, with that of type I being largely due to attenuation. In this region (results were shown at $P = 3{\cdot}08$), the *wave conversion* phenomenon was observed: wave type I vanished quickly, but leaked some of its energy to wave type II, which propagated more readily through the system.

3. STATISTICS OF LYAPUNOV EXPONENTS

We now extend the work of reference [16] to include a more systematic identification of frequency regions in which localization occurs. In addition, we suggest a statistical measure for predicting wave conversion.

In Figure 3, we consider the multi-span beam with $S = 0{\cdot}01$. The averages and the standard deviations of the Lyapunov exponents found at each iteration of the Wolf algorithm are shown. The standard deviation for the second Lyapunov exponent has a peak around $P = \pi$, which is the passband edge for wave type *II* in the ordered system. This is due to a numerical singularity at this frequency. If we ignore this, we see that the standard deviations are greatest in the frequency region around the double passband. This indicates that localization plays a prominent role in the decay here, as was concluded from a more costly analysis in reference [16]. Note that as P increases, the standard deviations suddenly drop. Thus, we might consider the frequency at which this drop occurs to be an upper bound for the frequency range featuring localization.

We now examine the mono-coupled example in order to consider the use of the standard deviation to separate damping and disorder effects. The average of the first Lyapunov exponent (the localization factor), as well as its standard deviation, are shown in Figure

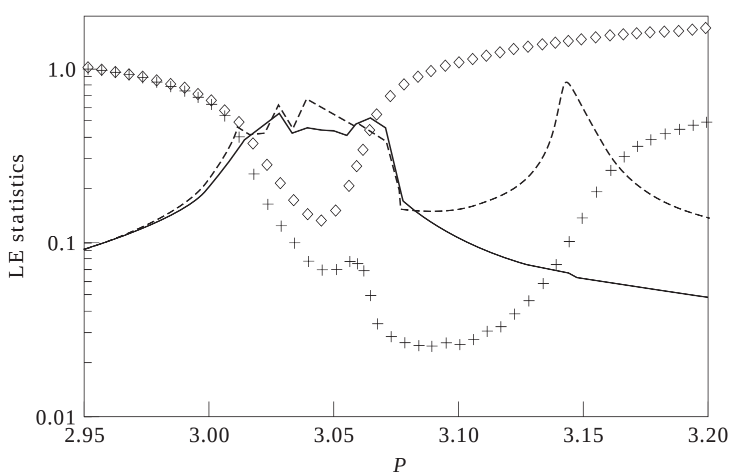

Figure 3. Lyapunov exponent statistics for the disordered multi-span beam, $\bar{K} = 50$, $S = 0{\cdot}01$, $N = 50\,000$ iterations: ——, mean of γ_1; - - - -, mean of γ_2; $\diamond$, standard deviation of γ_1; $+$, standard deviation of γ_2.

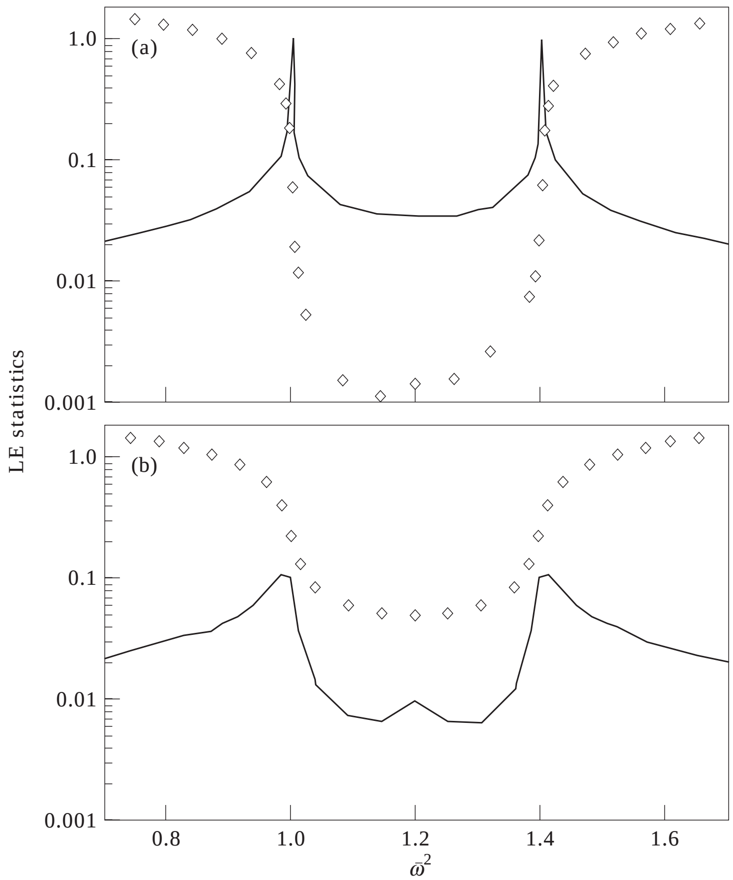

Figure 4. The statistics of the first Lyapunov exponent: ——, mean of γ_1; ◇, standard deviation of γ_1. (a) The *undamped*, disordered mono-coupled system: $R = 0{\cdot}1$, $S = 0{\cdot}01$, $N = 10\,000$ iterations. (b) The *damped*, disordered mono-coupled system: $R = 0{\cdot}1$, $S = 0{\cdot}01$, $\delta = 0{\cdot}01$, $N = 500$ iterations.

4(a) for the undamped, disordered system with a small disorder-to-coupling ratio. Again, we see peaks in the standard deviation at the passband edges due to numerical singularities. Otherwise, the standard deviation is greatest inside the passband region, and decreases outside (note the logarithmic scale). This decrease does not seem especially dramatic, unless we compare the standard deviation to the mean value. Inside the passband region, the standard deviation is more than an order of magnitude larger than the mean, while in the stopband regions the mean is more than an order of magnitude larger than the standard deviation. We quickly conclude that localization is confined to the passband region, which supports the observations from the detailed investigation of reference [21].

In Figure 4(b), we include structural damping in the same mono-coupled system. Now we see that the standard deviation *decreases* inside the passband. Outside the passband, where off-resonance effects are most significant, the standard deviation is similar to that of the undamped case. For all frequencies (ignoring the peaks at the passband edges), the mean is an order of magnitude greater than the standard deviation. This supports the conclusion of reference [21] that damping effects (versus disorder effects) dominate the decay rate for the small disorder-to-coupling ratio case.

We now turn our attention to the multi-span beam and the wave conversion phenomenon. We have seen that in the region of greatest localization, there was a mixing

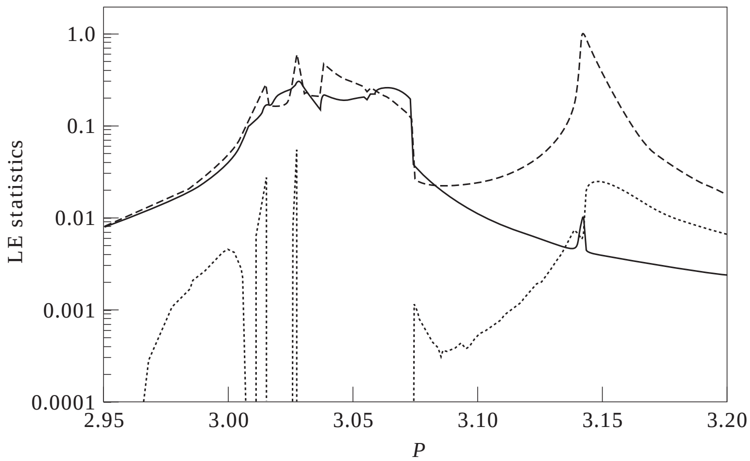

Figure 5. Lyapunov exponent statistics for the disordered beam with $\bar{K} = 50$, $S = 0{\cdot}01$, $N = 200\,000$ iterations: ——, variance of γ_1; ----, variance of γ_2; · · ·, covariance.

of the wave types [16]. Outside this region, we observed wave conversion, whereby one wave died quickly, but leaked energy to the other wave. We propose here that the covariance of the Lyapunov exponents may provide a statistical measure which is capable of predicting wave conversion. The covariance measures the statistical dependence of two random variables. If the covariance is zero, then these random variables are independent. Therefore, it seems that the covariance of two Lyapunov exponents would be greatest in regions of wave conversion, since here the decay rates are directly related, and that relationship is governed by the disorder effects.

In Figure 5, we show the covariance of the first and second Lyapunov exponents for the disordered multi-span beam. The frequencies at which we might predict wave conversion based on the covariance include $P = 3{\cdot}08$, where we previously observed the wave conversion phenomenon [16].

In general, we see that the non-trivial values of the covariance occur outside the frequency region in which the variances are greatest. This indicates that in different frequency regions, disorder has very different effects on the waves. In the frequency band in which the variances are highest, disorder causes a mixing of waves, but confines the energy of each. Outside this band, disorder allows a leakage of energy from the wave which decays most quickly due to off-resonance—the wave conversion phenomenon. In this case, by leaking energy to the wave that decays least, disorder allows a *greater* transmission of energy.

4. CONCLUSIONS

Lyapunov exponents (or localization factors) provide a measure of the spatial decay of the wave (or vibration) amplitudes in nearly periodic structures. It is of interest to identify the frequency regions in which the Lyapunov exponents indicate decay due to localization, as opposed to decay due to off-resonance or damping. This is because localization is a confinement of vibration energy that can lead to a drastic and undesirable increase in maximum response amplitudes.

Since numerical Lyapunov exponents and localization factors are *averaged* values computed from many iterations of an algorithm, more information can be obtained from other statistics besides the mean. It was found that the standard deviations of the

Lyapunov exponents determine frequency bands in which localization occurs. It was also suggested that the covariance of two Lyapunov exponents may predict the wave conversion phenomenon in specific frequency ranges. The Lyapunov exponent averages can then be used to compare the expected strength of localization at different frequencies. The systematic identification of critical frequency regions may be the most useful aspect of a Lyapunov exponent analysis.

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